

**NATIONAL BUREAU OF STANDARDS REPORT**

6397

Draft of  
Part II, section 4 (Sensitivity Testing)  
for  
MANUAL ON EXPERIMENTAL STATISTICS  
FOR ORDNANCE ENGINEERS

A Report to  
Office of Ordnance Research  
Department of the Army



**U. S. DEPARTMENT OF COMMERCE  
NATIONAL BUREAU OF STANDARDS**

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# NATIONAL BUREAU OF STANDARDS REPORT

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Draft of  
Part II, section 4 (Sensitivity Testing)

for  
MANUAL ON EXPERIMENTAL STATISTICS  
FOR ORDNANCE ENGINEERS

Prepared by  
Statistical Engineering Laboratory

A Report to  
Office of Ordnance Research  
Department of the Army

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## NOTICE

This report is a draft of Part II, section 4, covering techniques used in Sensitivity Testing. Table references are to Tables listed in NBS Report 5320.

No known inaccuracies exist in the present draft, but improvements in arrangement and exposition might be made at a later date.



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#### 4. Sensitivity Testing

##### 4.1 Experimental situation

The term "sensitivity test" is commonly applied to the following situation:

- (1) A test item will "respond" or "not respond" to a certain level of test stimulus (e.g., a shell will explode or not explode when subjected to a certain shock).
- (2) The test is "destructive" to the item being tested, no matter what the outcome of the test. Either the item is completely destroyed, or the characteristics of the item are so changed that further tests are meaningless.
- (3) The percentage of items expected to "respond" (fail, explode, die) increases as the severity of the test is increased.

In this general situation, there are variable (and usually controllable) levels of test which can be applied, e.g., height of drop in a shock test, dosage of a poison in tests of insecticides, etc. We assume that each object has an associated critical level or threshold value. If the test stimulus applied exceeds this critical level, the object responds (fails, explodes, dies). If the test applied does not exceed



this critical level, the object does not respond. For any particular object, the exact critical level cannot be determined. More than one object may be tested at a fixed critical level, however, and inferences may be made about the distribution of critical levels in a population of objects from which the tested samples came.

An ordnance example might involve the detonation of samples of an explosive in powder form by dropping a specified weight on them from various heights. If the weight is dropped from a height below the sample's critical level (in this case, the lowest height at which the weight will cause the sample to explode), the sample does not explode, but the powder may be packed more tightly than before and therefore the test cannot be repeated at increased height. If the weight is dropped from above the sample's critical level, the sample is destroyed.

Thus the experimenter obtains data of the following type: objects were tested at the  $k$  stimulus levels  $x_1, x_2, \dots, x_k$ . Of the  $n_i$  objects tested at level  $x_i$ ,  $r_i$  responded and  $n_i - r_i$  did not respond.



An ordnance example which will be used to illustrate the computational techniques is the following:

Example 4.1

Groups of fuzes are subjected to specified values of peak voltage. For each group we observe the number which fire in less than a specified time. The observations are summarized in Table 4.1.

Table 4.1

Peak Voltage	n	r	$p=r/n$
10.0	12	0	0
15.0	12	0	0
17.5	12	1	.08
20.0	13	2	.15
22.5	10	3	.30
25.0	13	6	.46
30.0	12	8	.67
35.0	13	9	.69
40.0	13	11	.85
50.0	11	10	.91
60.0	11	11	1.00

Several methods for collecting and analyzing such data are described and illustrated in this section. Sections 4.2 and 4.3 detail methods of analysis for the usual testing situation, where the levels of test are assigned before the test begins. 4.4 details a special method applicable when the





test levels can be different for each successive object tested and can be easily changed during the course of the experiment. 4.5 discusses the situation where the test levels cannot be completely controlled.

See:

- 4.2 Karber method of analysis - for a simple method not requiring the assumption of normality, which gives good estimates of the mean and standard deviation of the distribution of critical levels.
- 4.3 Probit method of analysis - for a method which assumes normal distribution of critical levels and permits estimation of percentage points of the distribution of critical levels.
  - 4.3.1 Graphical probit solution
  - 4.3.2 Exact probit solution
  - 4.3.3 Method of testing whether or not the line is an adequate representation of the data
- 4.4 Up and down design - for a method in which successive test levels are determined on the basis of the outcome of immediately preceding tests; assuming normal distribution, this method gives very accurate estimates of the mean.
- 4.5 Methods of sensitivity testing when the level of stimulus cannot be controlled.

Most of these methods involve assumptions about the distrib-



ution of the critical levels such as, for example, that it is "normal". Frequently the critical levels as measured in the original units (or "natural" units) do not meet these assumptions, but there exists a transformation such that the distribution of the transformed values does. The logarithm of the original value is perhaps the most frequently used transformation.

When a transformation is used, all comments on the selection of testing levels and all computational instructions refer to the transformed values, not the original ones. Usually, however, it is desirable to state the final results of the analysis in terms of the original units. For most transformations (including the logarithmic), the percentile estimates\* and their associated confidence intervals are easily converted into the original units. Suppose that the stimulus levels are originally measured in "y" units, and transformed values, e.g.,  $x = \log y$ , are used in the computations. If  $\hat{x}_p$  is an estimate of  $x_p$ , as here defined\*, and  $[a, b]$  is a  $1-\alpha$  confidence interval estimate of  $x_p$ , then  $\text{antilog } \hat{x}_p$  and

---

\*) For any random variable X, the "(100 p)th percentile" of its distribution is the value  $x_p$  such that the probability that X is  $\leq x_p$  is equal to p.



[antilog a, antilog b] give respectively a point estimate and a  $1-\alpha$  confidence interval estimate of  $y_p$ , the  $(100 p)$ th percentile of the distribution of  $y$  values. This relationship does not hold for the means and standard deviations of the distributions. If  $m$  and  $s$  are estimates of the mean and standard deviation of the distribution of the  $x$  values, antilog  $m$  and antilog  $s$  should not be considered estimates of the mean and standard deviation of the distributions of  $y$ 's.

**NOTE:**

In this section, "normal" means that the expected proportion of items responding at a stimulus level  $x$  is

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{- (x-\mu)^2 / 2\sigma^2} dx$$



4.2 Kärber method of analysis - a simple method of analysis which does not require assumption of normality and which gives good estimates of mean and standard deviation.

This method gives simple nonparametric estimates of the mean and standard deviation of the distribution of critical levels. There are three situations when this method might be chosen: (1) the shape or mathematical form of the distribution is unknown; (2) quick and easy procedures for routine laboratory calculations are desired; and (3) good "initial estimates" or "first approximations" are needed for iterative computational procedures, for example, for the "exact probit solution" in Section 4.3. The Kärber method given below provides very good estimates of the mean and standard deviation of the distribution of critical levels in most laboratory situations. (For further discussion of the Kärber method, see [3].) It must be remembered that the Kärber method gives an estimate of the mean of the distribution; the mean of the distribution is not equal to the 50th percentile (that level  $x_0$  such that half the objects have critical levels  $< x_0$  and half have critical levels  $> x_0$ ) unless it is known that the distribution is symmetrical about its mean.





4.2.1 Selection of stimulus levels: Order the stimulus levels to be used in the test by their magnitude - in other words, let  $x_1 < x_2 < \dots < x_k$ . For the Kärber method to be used,  $x_1$  must be sufficiently low so that there are no responses among the objects tested ( $r_1 = 0$ ) and  $x_k$  must be sufficiently high that all objects tested respond ( $r_k = n_k$ ). In other words,  $x_1$  and  $x_k$  are to be chosen so that they are likely to cover the entire range of critical levels in the population. In addition, it is preferable to have more (and consequently, more closely spaced) test levels with fewer objects tested at each level than to have only a few test levels and a large number of objects at each level.



**4.2.2 Solution:** Prepare a table with 9 columns headed  $x$ ,  $n$ ,  $r$ ,  $p$ ,  $(p_{i+1} - p_i)$ ,  $a$ ,  $a^2$ ,  $d$ ,  $d^2$ . (See Table 4.2.3)

- (1) In column  $x$ , enter the stimulus levels used in the test from lowest to highest - that is, enter  $x_1, x_2, \dots, x_k$  where  $x_1 < x_2 < \dots < x_k$ .
- (2) In columns  $n$  and  $r$ , for each  $x_i$ , enter the corresponding  $n_i$  (number of objects tested at that level) and  $r_i$  (number of responses).
- (3) Corresponding to each  $x$ , compute  $p_i = \frac{r_i}{n_i}$  and enter this in column  $p$ . (Remember that  $p_1 = 0$  and  $p_k = 1$  if this solution is to be used.)
- (4) Corresponding to each  $x_i$  (except  $x_k$ ), compute  $p_{i+1} - p_i$  and enter it in column  $(p_{i+1} - p_i)$ . There is no entry in this column corresponding to  $x_k$ .
- (5) Corresponding to each  $x_i$  (except  $x_k$ ), compute  $a_i = \frac{x_{i+1} + x_i}{2}$ , the midpoint of the interval from  $x_i$  to  $x_{i+1}$ . Tabulate the  $a_i$  values in column  $a$ . There is no entry in this column corresponding to  $x_k$ .
- (6) Corresponding to each  $a_i$ , enter  $a_i^2$  in column  $a^2$ .
- (7) Corresponding to each  $x_i$  (except  $x_k$ ), compute  $d_i = x_{i+1} - x_i$ , the length of the interval from  $x_i$  to  $x_{i+1}$ . Tabulate the  $d_i$  values in column  $d$ . There is no entry in this column corresponding to  $x_k$ .
- (8) Corresponding to each  $d_i$ , enter  $d_i^2$  in column  $d^2$ .



(9) Compute  $m = \sum (p_{i+1} - p_i) a_i$ , the sum of products of corresponding entries in the 5th and 6th columns.  $m$  is our estimate of the mean of the distribution of critical levels.

(10)  $S_1 = \sum (p_{i+1} - p_i) a_i^2$ , the sum of products of corresponding entries in the 5th and 7th columns.  
 $S_2 = \sum (p_{i+1} - p_i) d_i^2$ , the sum of products of corresponding entries in the 5th and 9th columns.

$$s^2 = S_1 - m^2 - \frac{S_2}{12}.$$

$s$  is our estimate of the standard deviation of the distribution of critical levels.



#### 4.2.3 Example:

The data of Table 4.1 are used to illustrate the solution. In this illustration we have assumed that nothing is known about the distribution of critical levels, and the computations are done in natural units. If the distribution of critical levels were known or presumed to be log normal, and if the Kärber method were being used to provide a quick and easy answer, or to provide initial estimates for the exact probit solution (4.3), then the computations would have been done on the transformed values  $x = \log \text{ peak voltage}$ .

The entries and calculations of Steps (1) through (10) of the solution are shown in Table 4.2.3





Table 4.2.3 Kärber Method of Analysis for Fuze Data

Level of stimulus, peak voltage	Number of objects tested	Number of objects responding	$r/n$	$p_{i+1}-p_i$	$a$	$a^2$	$d$	$d^2$
$x$	$n$	$r$	$p$	(4)	(5)	(6)	(7)	(8)
Steps: (1)	(2)	(2)	(3)					
10.0	12	0	0	0	12.5	156.25	5.0	25.
15.0	12	0	0	.08	16.25	264.06	2.5	6.25
17.5	12	1	.08	.07	18.75	351.56	2.5	6.25
20.0	13	2	.15	.15	21.25	451.56	2.5	6.25
22.5	10	3	.30	.16	23.75	564.06	2.5	6.25
25.0	13	6	.46	.21	27.50	756.25	5.0	25.
30.0	12	8	.67	.02	32.5	1056.25	5.0	25.
35.0	13	9	.69	.16	37.5	1406.25	5.0	25.
40.0	13	11	.85	.06	45.0	2025.	10.0	100.
50.0	11	10	.91	.09	55.0	3025.	10.0	100.
60.0	11	11	1.00					

(12)



Table 4.2.3 continued:

Step(9)	$m = \Sigma(p_{i+1} - p_i) a_i$	Step(10)	$S_1 = \Sigma(p_{i+1} - p_i)^2 a_i^2 = 1002.4051$
pv	$m = 29.68 = \text{mean peak voltage}$		$S_2 = \Sigma(p_{i+1} - p_i)^2 d^2 = 27.6250$
			$s^2 = S_1 - m^2 - \frac{S_2}{12}$
			$= 1002.4051 - 880.9024 - 2.3021$
			$s^2 = 119.2006$
			$s = 10.92 = \text{estimated standard deviation of peak voltage}$

(12a)



4.2.4 Simplified calculation applicable when test levels (x's) are equally spaced and equal numbers of items are tested at each level

- (1) Prepare a table with columns headed  
 $x$ ,  $r$ ,  $p$ , cumulative  $p$ .
- (2) In the  $x$  column, enter the test levels from lowest to highest, i.e. enter  $x_1, x_2, \dots, x_k$  where  $x_1 < x_2 \dots < x_k$
- (3) Enter  $r_i$ , the number of objects responding at each  $x_i$ .
- (4) Corresponding to each  $x_i$  compute  $p_i = r_i/n$ .  
 (n is the number of objects tested at each level and is the same for all levels).  
 Remember that  $p_1$  must equal 0 and  $p_k$  must equal 1 for this solution to be used.
- (5) In the last column, enter the cumulative  $p$ , i.e. at  $x_i$ , the sum of all  $p$  up to and including  $p_i$
- (6) Let  $x_k =$  highest test level  
 $d =$  interval between successive test levels  
 $S_1 =$  sum of column  $p$   
 $s_2 =$  sum of column "cumulative  $p$ "



(14)

$$\text{Step (7)} \quad m = x_k - d(S_1 - 1/2)$$

$$\text{Step (8)} \quad s^2 = d^2(2S_2 - S_1 - S_1^2 - 1/12)$$

### Example

In order to demonstrate the computing procedure, the fuze data of Table 4.1 have been arbitrarily changed to have equal spacing in  $x$  and equal numbers of objects at each level. Assume that 12 objects were tested at each level and that the responses were as shown in Table 4.2.4.





Table 4.2.4 Special Case of Kärber Method of Analysis

Peak Voltage $x$ (Step 2)	Number of objects responding $r$ (Step 3)	Proportion of objects responding $p = r/12$ (Step 4)	Cumulative proportion (Step 5)
10.0	0	0	0
15.0	0	0	0
20.0	2	.17	.17
25.0	6	.50	.67
30.0	8	.67	1.34
35.0	9	.75	2.09
40.0	10	.83	2.92
45.0	10	.83	3.75
50.0	11	.92	4.67
55.0	12	<u>1.00</u>	<u>5.67</u>
		$S_1 = 5.67$	$S_2 = 21.28$

(15)

Step(6)  $x_k =$  highest test level = 55

$d =$  interval in  $x = 5$

$S_1 =$  sum of  $p = 5.67$

$S_2 =$  sum of "cum.p" = 21.28

Step(7)  $m = x_k - d(S_1 - 1/2)$

$= 55 - 5(5.67 - .50)$

$m = 29.15 =$  mean peak voltage

Step(8)  $s^2 = d^2(2S_2 - S_1 - S_1^2 - 1/12)$

$= 25(42.56 - 5.67 - 32.15 - .08)$

$s^2 = 116.50$

$s = 10.79 =$  estimated standard deviation of peak voltage



#### 4.3 Probit method of analysis

When it is assumed that the distribution of critical levels is of a particular type, methods for estimating the properties of the distribution are tailored to it.

There are three types of distributions for sensitivity data that have been studied extensively in the statistical literature - the "normal", "logistic", and the "angular".

For the logistic, see [1] and [8]; for the angular see [9]. Only the "normal" will be discussed in detail here because it is most frequently used in ordnance sensitivity testing. When the stimulus levels used in the test are between the levels which cut off the lower and upper 10% of the distribution (most testing is performed in this range), any one of these types will fit the data nearly as well (or as poorly) as another, no matter what the true distribution of critical levels is. However, estimates of the parameters of the "logistic" or the "angular" distributions involve simpler computations than are given below for the "normal"; for example, see the technique described in [8] for the logistic curve.

The procedures described in this section assume that the distribution of critical values is "normal" - that is, for all  $x$  the proportion of objects which have critical levels between  $x$  and  $x+dx$  is equal to the area of some



"normal" curve between  $x$  and  $x+dx$ . In general, the procedures will not be too sensitive to moderate departures from normality provided one does not extrapolate beyond the range of the data.

Problem 4.3 -  $k$  different levels  $x_1, x_2, \dots, x_k$  of a stimulus are applied to  $n_1, n_2, \dots, n_k$  objects, with  $r_1, r_2, \dots, r_k$  responses, respectively. Let  $p_i = \frac{r_i}{n_i}$ .

- (a) At what level  $\underline{m}$  of the stimulus would half of similar objects be expected to respond? (Or, equivalently under the assumption of normality, what is the mean of the critical levels of all such objects?)
- (b) Estimate the relation between the level of the stimulus and the proportion of objects responding.

Selection of stimulus levels: There are no simple cut-and-dried rules. A general guide can be given in terms of the purpose of the experiment:

- (1) If the experimenter is interested in estimating a specific percentage point, the stimulus levels to be used in the test should be fairly close to that point and should bracket the point. It is pertinent here to emphasize that extrapolations



may lead to serious error, particularly if the experimenter attempts to estimate an extreme percentage point (the 5% or 99% point, say) from observations at stimulus levels which all lie to one side of that point.

- (2) The test levels should cover a sufficiently wide range so that the proportion responding ( $p_i$ ) varies from near 0 to near 1, if:

- (a) One is interested in the relation between stimulus level and percentage response over the entire range or
- (b) One is interested in estimating the standard deviation of the distribution of critical levels (or equivalently the slope of the regression line in the probit solution) or
- (c) One is interested in testing the assumption of "normality".

#### Basis of the method

If the critical levels are normally distributed (with unknown mean  $\mu$  and standard deviation  $\sigma$ ), then  $Y_i'$ , which can be determined from tables of the normal integral, is a linear function of the corresponding stimulus level  $x_i$ .





(19')

$Y_i'$  is determined from:  
 $Y_i' - 5$

$$p_i = \int_{-\infty}^{Y_i' - 5} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy .$$

The "5" in the upper limit of the integral is introduced for computational convenience.

Least squares procedures are used to estimate the best straight line passing through the  $k$  points  $(Y_i', x_i)$ . The formulas take account of the fact that the points do not have equal weights. The line will be expressed as

$$Y_p = 5 + b(x - m)$$

where  $m$  and  $b$  are estimated from the data,  $x$  is the stimulus level, and  $Y_p$  is related to  $p$ , the probability that an object's critical level is  $\leq x$ , by the formula

$$\int_{-\infty}^{Y_p - 5} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = p .$$

$m$  is an estimate of  $\mu$  (the mean and 50th percentile of the underlying "normal" distribution) and  $b$  is an estimate of  $1/\sigma$  (the reciprocal of the standard deviation of the critical values).



Solutions described

We shall describe two methods of solution: the graphical probit method (4.3.1) and the computational probit method (4.3.2). The graphical method is much simpler, and is sufficiently precise for many purposes.



#### 4.3.1 Graphical probit solution.

To facilitate the calculations, prepare a table with 9 columns, headed respectively  $x$ ,  $n$ ,  $r$ ,  $p$ ,  $Y'$ ,  $Y$ ,  $W$ ,  $nW$ ,  $nWx$ . (See Table 4.3.1)

- (1) In column 1, enter the levels  $x_1, x_2, \dots, x_k$  used in the experiment.
- (2) In columns  $n$  and  $r$ , record the values of  $n_i$  and  $r_i$ , corresponding to  $x_i$ .
- (3) In column  $p$  compute the respective proportions responding,  $p_i = r_i/n_i$ .
- (4) Use Table Ib to obtain  $z_p$  corresponding to  $\bar{p}$ . In column  $Y'$ , enter  $Y' = z_p + 5$  corresponding to each  $p$ .
- (5) Plot  $Y'$  as ordinate against  $x$  as abscissa on ordinary rectangular coordinate graph paper. See Figure 4.3.1. If probit paper\* is available, it can be used and the column  $Y'$  is omitted from the table. The percentages responding at each  $x$  (% response =  $100 p_i$ ) are plotted on probit paper using the left vertical scale; the right vertical scale gives the corresponding  $z_p$ , and hence the corresponding  $Y-5$ .

---

\* "Probit paper" is also called "normal deviate paper", "normal ruling", etc.



- (6) Whichever graph paper is used, draw a straight line by eye to fit the  $k$  points. Only vertical deviations from the line are to be considered in fitting, and points for which the value of  $Y'$  is outside the interval 2.5-7.5 may almost be disregarded unless for those points  $n_i$  is much larger than for those points inside the interval. (Points outside this interval are beyond the range of the probit paper).
- (7) For each value of  $x$  plotted on the graph, read the ordinate  $Y$  of the line (on probit scale if on probit paper) and record the values in the  $Y$  column of the table.

Before proceeding further, it is often desirable to test to see if the line is an adequate representation of the data. The procedure given in 4.3.3. may be used.

- (8) Calculate  $b$ , the slope of the fitted line, as the increase in  $Y$  for a unit increase in  $x$ . Mark 2 convenient points,  $c$  and  $d$  on the line. Read off the corresponding values for  $Y$ .





$$x_c =$$

$$Y_c =$$

$$x_d =$$

$$Y_d =$$

Then  $b = \frac{Y_d - Y_c}{x_d - x_c}$  . Read off  $m$ , the value of  $x$

corresponding to  $Y = 5$  . (Probit = 5, on probit paper). We may then write the equation of the line as  $Y = 5 + b(x - m)$  .

- (9) The relation between a given level of stimulus ( $x'$ ) and the proportion of individuals responding ( $p'$ ) is estimated by the relationship  $z_{p'} = b(x' - m)$  where the  $p'$  corresponding to  $z_{p'}$  is given in Table Ia. (Or, this relationship can be read directly from the straight line drawn on probit paper.)

$m$  is our estimate of  $\mu$ , the stimulus level at which we would expect half of similar individuals to respond. A  $1-\alpha$  confidence interval estimate of  $\mu$  may be computed as follows:

- (10) Look up  $z_{1-\alpha/2}$  in Table Ib.
- (11) In the column  $W$ , corresponding to each value of  $Y$ , enter the value of  $W$  obtained from Table VIII.
- (12) Corresponding to each value of  $x$ , compute  $nW$  and  $nWx$  and enter then in the last columns of the table.



(24)

- (13) Compute  $\sum n_i W_i$  and  $\sum n_i W_i x_i$ , the totals for the last two columns.
- (14) Compute  $s_m^2 = \frac{1}{b^2} \left( \frac{1}{\sum n W} \right)$ , the estimate of the variance of  $m$ . The estimate is slightly too small on the average, but this bias is negligible provided that  $\bar{x} = \sum n W x / \sum n W$  is approximately equal to  $m$ . If the two differ considerably, then the quantity  $S_{xx} = \sum n_i W_i x_i^2 - (\sum n W x)^2 / \sum n W$  should be computed, and our estimate of the variance of  $m$  becomes

$$s_m^2 = \frac{1}{b^2} \frac{1}{\sum n W} + \frac{(m - \bar{x})^2}{S_{xx}} .$$

- (15) A  $1-\alpha$  confidence interval estimate of  $\mu$  is the interval from  $m - z_{1-\alpha/2} s_m$  to  $m + z_{1-\alpha/2} s_m$ .



Example

The data of Table 4.1 were plotted on normal probability paper using both  $x = \text{peak voltage}$  and  $x = \log_{10} \text{peak voltage}$ , as abscissa. The data plotted more nearly as a straight line using  $x = \log_{10} \text{peak voltage}$ , and therefore all of the probit method calculations are done on the transformed variable.

See Table 4.3.1.



Table 4.3.1 Graphical Probit Solution

Level of stimulus = $\log_{10}$ Peak Voltage	Number of objects tested		Number of objects responding		$r/n$		Y'	Y	W		nW		nWx	
	Step(1)	Step(2)	Step(2)	Step(2)	Step(3)	Step(4)	Step(7)	Step(11)	Step(12)	Step(12)	Step(12)	Step(12)	Step(12)	Step(12)
	1.0000	12	0	0	0									
	1.1761	12	0	0	0									
	1.2430	12	1	1	.08	3.59	3.7	.336	4.032	4.032	5.012	5.012		
	1.3010	13	2	2	.15	3.96	4.1	.471	6.123	6.123	7.966	7.966		
	1.3522	10	3	3	.30	4.48	4.5	.581	5.810	5.810	7.856	7.856		
	1.3979	13	6	6	.46	4.90	4.8	.627	8.151	8.151	11.394	11.394		
	1.4771	12	8	8	.67	5.44	5.3	.616	7.392	7.392	10.919	10.919		
	1.5441	13	9	9	.69	5.50	5.8	.503	6.539	6.539	10.097	10.097		
	1.6021	13	11	11	.85	6.04	6.2	.370	4.810	4.810	7.706	7.706		
	1.6990	11	10	10	.91	6.34	6.9	.154	1.694	1.694	2.878	2.878		
	1.7782	11	11	11	1.00		7.4	.062	0.682	0.682	1.213	1.213		

$\Sigma nW = 45.233$

$\Sigma nWx = 65.041$





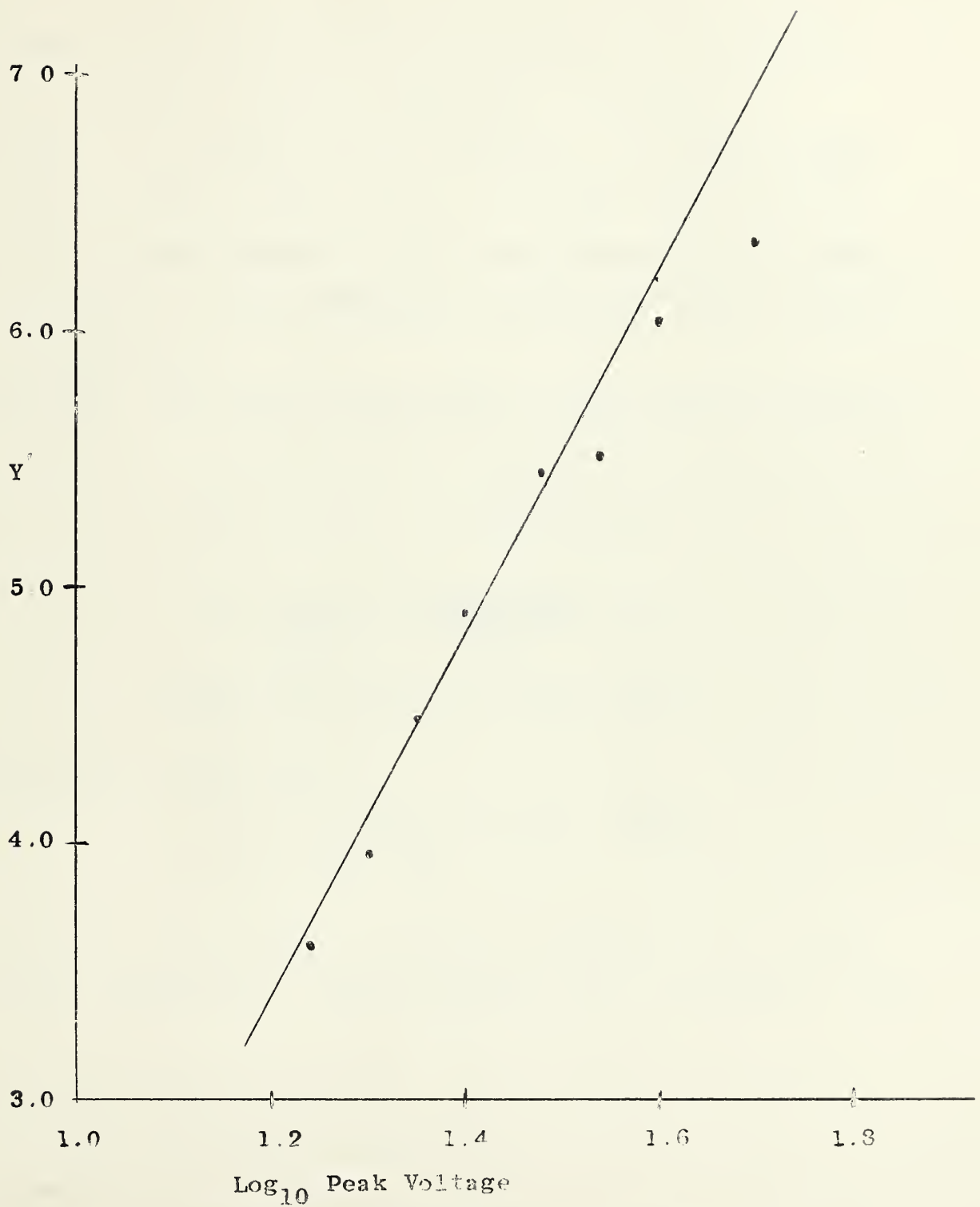


Fig. 4.3.1 PROBIT REGRESSION LINE (FITTED BY EYE)



Steps (1) through (4) of the Solution result in filling in the first 5 columns of Table 4.3.1 (through  $Y'$ )

Steps (5) through (7): Plot  $Y'$  against  $x$  on ordinary graph paper as shown in Fig. 4.3.1. A straight line is fitted to the plotted points by eye. The ordinate  $Y$  of the line is read off at each observed  $x$  and entered in the  $Y$  column of Table 4.3.1.

Step (8) The slope of the line  $b$  is calculated as follows:

$$x_c = 1.4 \qquad Y_c = 4.77$$

$$x_d = 1.5 \qquad Y_d = 5.47$$

$$b = \frac{Y_d - Y_c}{x_d - x_c} = \frac{5.47 - 4.77}{0.1} = 7.0$$

$$m \text{ (the value of } x \text{ at } Y = 5) = 1.43$$

Equation of the line:

$$Y = 5 + b(x-m) = 5 + 7x - 10.01$$

$$Y = -5.01 + 7x$$

Step (9) If we wish to estimate the proportion of individuals responding at a peak voltage = 18, for example, then:



$$x' = \log_{10} 18 = 1.255$$

$$\begin{aligned} z_{p'} &= b(x' - m) = 7(1.255 - 1.43) \\ &= 8.785 - 10.01 \end{aligned}$$

$$z_{p'} = -1.22$$

$P = .11$ , the proportion which may be expected to respond at  $PV = 18$

Steps (10) through (15)

A 95% confidence interval estimate of  $\mu$  (the level at which we would expect half of the individuals to respond) is obtained as follows:

Columns W, nW and nWx are computed and entered in Table 4.3.1

$$\Sigma nW = 45.233$$

$$\Sigma nWx = 65.041$$

$$s_m^2 = \frac{1}{b^2} \frac{1}{\Sigma nW} = \frac{1}{49} \left( \frac{1}{45.233} \right) = \frac{1}{2216.417}$$

$$s_m^2 = .000451 \qquad s = .0212$$

(since  $\bar{x} = 1.44$ , approximately equal to  $m = 1.43$ , we do not bother to use the more complicated formula for  $s_m^2$ .)



(30)

Let  $\alpha = .05$  ,  $z_{1-\alpha/2} = 1.96$

A 95% confidence interval estimate of  $\mu$  is the interval  
1.39 to 1.47

$$(\bar{m} \pm 1.96 s_m = 1.43 \pm .042)$$





#### 4.3.2 Exact probit solution.

The graphical solution is very often adequate, but complications in the data may make this solution less satisfactory in some cases and an arithmetic technique may be necessary. For example, the points may be too irregular for any confidence to be placed in a line drawn by eye, or the weights ( $nW$ ) that should be attached to each point may be so different as to make it difficult to visually adjust for them.

The probit solution given here involves a series of successive approximations, the first of which is given by the graphical probit solution or by the Kärber method described in Section 4.2. For the exact solution we need a table with 10 columns headed respectively  $x$ ,  $n$ ,  $r$ ,  $p$ ,  $Y$ ,  $W$ ,  $nW$ ,  $nWx$ ,  $y$ ,  $nWy$ . (See Table 4.3.2.1).

- (1) In column 1, enter the levels  $x_1, x_2, \dots, x_k$  used in the experiment.
- (2) In columns  $n$  and  $r$ , record the values of  $n_i$  and  $r_i$ , corresponding to  $x_i$ .
- (3) In column  $p$  compute the respective proportions responding,  $p_i = r_i/n_i$ .



- (4) Obtain values for column Y by either method (a) or (b) below:

- (a) Follow instructions (1) through (7) of the graphical probit solution to obtain the Y-values. These are then tabulated in the table for the exact solution.
- (b) Follow instructions (1) through (10) of the Kärber method described in section 4.2 to compute  $m$  and  $s$ . For initial estimates, take  $m_0 = m$ ,  $b_0 = \frac{1}{s}$  and corresponding to each  $x_i$ , use the equation

$$Y_i = 5 + b_0 (x_i - m_0)$$

to compute the values for the Y column of the table for the exact solution.

Unless  $n$  is very large, 1 decimal in Y is sufficient. If 2 decimals in Y should be required, the Tables in this Manual are not convenient; consult [6] for more extensive tables.

- (5) In the column W, corresponding to each value of Y, enter the value of W obtained from Table VIII.



(6) Corresponding to each value of  $x$ , compute  $n_i W_i$  and  $n_i W_i x_i$  and enter them in the  $nW$  and  $nWx$  columns respectively of the table.

(7) Corresponding to each "expected probit"  $Y$  in the  $Y$  column, Table IX is used to compute the "working probit" ,  $y$ , as follows:

(a) For  $Y \leq 5.0$ ,  $y = y_0 + p\left(\frac{1}{Z}\right)$

(b) For  $Y \geq 5.0$ ,  $y = y_{100} - (1-p)\left(\frac{1}{Z}\right)$

where for each  $Y$ ,  $p$  is the corresponding entry in the  $p$  column of the table, and Table IX gives the values for  $y_0$  (or  $y_{100}$ ) and  $\frac{1}{Z}$ . Tabulate the values in the column  $y$ .

(8) For each value of  $y$ , calculate the value  $n_i W_i y_i$  and enter it in column  $nWy$ .

(9) Compute

$\Sigma n_i W_i$  , the sum of column  $nW$  .

$\Sigma n_i W_i x_i$  , the sum of column  $nWx$  .

$\Sigma n_i W_i y_i$  , the sum of column  $nWy$  .

$S_1 = \Sigma n_i W_i x_i^2$  ; the sum of the products of corresponding elements in columns  $x$  and  $nWx$ .



(34)

$S_2 = \sum n_i W_i x_i y_i$  , the sum of products of corresponding elements in columns nWx and y .

$$S_3 = \sum nW y^2$$

$$S_{xx} = S_1 - (\sum nWx)^2 / \sum nW$$

$$S_{xy} = S_2 - (\sum nWx)(\sum nWy) / \sum nW$$

$$S_{yy} = S_3 - (\sum nWy)^2 / \sum nW$$

- (10) Compute  
     $b = S_{xy} / S_{xx}$   
     $\bar{x} = \sum nWx / \sum nW$  ,  $\bar{y} = \sum nWy / \sum nW$

The equation of the probit regression line is

$$Y = \bar{y} + b(x - \bar{x}) .$$

The procedure for obtaining the "best" line is an iterative one, and in theory one should repeat the above process until the same equation is obtained for two successive iterations. Practically, one may often be able to see that an additional iteration will not materially change the equation.

The procedure for obtaining an additional iteration is as follows: Make a table with 9 columns headed x, n, p, Y, W, nW, nWx, y, nWy. (See Table 4.3.2.2)





- (11) Copy the first three columns,  $x$ ,  $n$ ,  $p$  from the previous table.
- (12) For each value of  $x$ , compute the corresponding value of  $Y$  using the equation  $Y = \bar{y} + b(x - \bar{x})$  calculated in the previous iteration.
- (13) Follow instructions (5) through (10).

To test whether the line is a good fit to the data, use the test procedure outlined in Section 4.3.3, using the  $\bar{y}$ ,  $\bar{x}$ , and  $b$  obtained from the last iteration performed to compute a new  $Y = \bar{y} + b(x - \bar{x})$  for each value of  $x$  used in the test.

#### Example

The fuze data is again used for illustration, and the line fit graphically in 4.3.1 is used as the first approximation.

Steps (1) through (3) consist of filling in the first 4 columns of Table 4.3.2.1

See Table 4.3.2.1.



Table 4.3.2.1 Exact Probit Solution

log <sub>10</sub> PV	=x	No. of objects		r	No. of objects responding		p	Expected Probit		Y	W	nW	nWx	y	nWy
		Step 1	Step 2		Step 2	Step 3									
		1.0000	12	0	0	0	2.0	.015	0.180	0.180	1.695	0.180	0.305		
		1.1761	12	0	0	0	3.2	.180	2.160	2.540	2.745	2.540	5.929		
		1.2430	12	1	.08	.15	3.7	.336	4.032	5.012	3.602	5.012	14.523		
		1.3010	13	2	.30	.46	4.1	.471	6.123	7.966	3.972	7.966	24.321		
		1.3522	10	3	.67	.85	4.5	.581	5.810	7.856	4.476	7.856	26.006		
		1.3979	13	6	.91	1.00	4.8	.627	8.151	11.394	4.900	11.394	39.940		
		1.4771	12	8			5.3	.616	7.392	10.919	5.436	10.919	40.183		
		1.5441	13	9			5.8	.503	6.539	10.097	5.461	10.097	35.709		(36)
		1.6021	13	11			6.2	.370	4.810	7.706	6.020	7.706	28.956		
		1.6990	11	10			6.9	.154	1.694	2.878	5.966	2.878	10.106		
		1.7782	11	11			7.4	.062	0.682	1.213	7.766	1.213	5.296		

$$\begin{aligned} \Sigma nW &= 47.573 \\ \Sigma nWx &= 67.761 \\ \Sigma nWy &= 231.274 \end{aligned}$$

$$\begin{aligned} S_1 &= \Sigma nWx^2 = 97.4232 \\ S_2^1 &= \Sigma nWxy = 335.4415 \end{aligned}$$

$$\begin{aligned} S_{xx} &= 0.9073 \\ S_{xy} &= 6.0244 \end{aligned}$$

$$\begin{aligned} \bar{x} &= 1.4244 \\ \bar{y} &= 4.8615 \\ b &= S_{xy}/S_{xx} = 6.640 \\ Y &= \bar{y} + b(x-\bar{x}) = -4.5965 + 6.640x \end{aligned}$$



Step (4) The Y values entered are copied from the Y column of the graphical probit solution (Table 4.3.1). These are the ordinates of the line at each observed x.

Step (5) through (8) fill in the remaining columns of Table 4.3.2.1

$$\text{Step (9)} \quad \Sigma nW = 47.573$$

$$\Sigma nWx = 67.761$$

$$\Sigma nWy = 231.274$$

$$S_1 = \Sigma nWx^2 = 97.4232$$

$$S_2 = \Sigma nWxy = 335.4415$$

$$S_{xx} = S_1 - (\Sigma nWx)^2 / \Sigma nW = 0.9073$$

$$S_{xy} = S_2 - (\Sigma nWx)(\Sigma nWy) / \Sigma nW = 6.0244$$

$$\text{Step (10)} \quad \bar{x} = \Sigma nWx / \Sigma nW = 1.4244$$

$$\bar{y} = \Sigma nWy / \Sigma nW = 4.8615$$

$$b = S_{xy} / S_{xx} = 6.640$$

The equation of the probit regression line is

$$Y = \bar{y} + b(x - \bar{x})$$

$$Y = -4.5965 + 6.640 x$$



Next iteration

An additional iteration is shown in Table 4.3.2.2. The first 3 columns (x, n, p) of Table 4.3.2.2 are copied from Table 4.3.2.1. The Y column is calculated by substituting observed values of x in the equation shown in Step (10) above. The remaining columns of Table 4.3.2.2 are filled in as described in Steps (5) through (8) of the Procedure.

$$\text{Step (9)} \quad \Sigma nW = 49.114$$

$$\Sigma nWx = 70.519$$

$$\Sigma nWy = 242.771$$

$$S_1 = \Sigma nWx^2 = 102.2663$$

$$S_2 = \Sigma nWxy = 355.3694$$

$$S_3 = \Sigma nWy^2 = 1247.7433$$

$$S_{xx} = 1.0135$$

$$S_{xy} = 6.7933$$

$$S_{yy} = 47.7238$$

$$\text{Step (10)} \quad \bar{x} = \Sigma nWx / \Sigma nW = 1.4358$$

$$\bar{y} = \Sigma nWy / \Sigma nW = 4.943$$

$$b = S_{xy} / S_{xx} = 6.703$$





The equation of the probit regression line is:

$$Y = \bar{y} + b(x - \bar{x})$$

$$Y = -4.6812 + 6.703 x$$

Using this equation to calculate values of Y, the Y values obtained differ very little from those obtained on the first iteration, and no further iterations are considered necessary.



Table 4.3.2.2 Exact Probit Solution (Second Iteration)

$\log_{10} PV$	No. of objects tested	$r/n$	$p$	Y	W	nW	nWx	y	nWy
1.0000	12	0	2.0	.015	.180	.180	1.695	.305	
1.1761	12	0	3.2	.180	2.160	2.540	2.745	5.929	
1.2430	12	.08	3.7	.336	4.032	5.012	3.602	14.523	
1.3010	13	.15	4.0	.439	5.707	7.425	3.964	22.623	
1.3522	10	.30	4.4	.558	5.580	7.545	4.477	24.982	
1.3979	13	.46	4.7	.616	8.008	11.194	4.904	39.271	
1.4771	12	.67	5.2	.627	7.524	11.114	5.432	40.870	
1.5441	13	.69	5.7	.532	6.916	10.679	5.482	37.914	
1.6021	13	.85	6.0	.439	5.707	9.143	6.036	34.447	
1.6990	11	.91	6.7	.208	2.288	3.887	6.217	14.224	
1.7782	11	1.00	7.2	.092	1.012	1.800	7.592	7.683	

(40)

$$\Sigma nW = 49.114 \quad \bar{x} = 1.4358$$

$$\Sigma nWx = 70.519 \quad \bar{y} = 4.943$$

$$\Sigma nWy = 242.771$$

$$S_1 = \Sigma nWx^2 = 102.2663 \quad b = S_{xy}/S_{xx} = 6.703$$

$$S_2 = \Sigma nWxy = 355.3694$$

$$S_3 = \Sigma nWy^2 = 1247.7433$$

$$S_{xx} = 1.0135$$

$$S_{xy} = 6.7933$$

$$S_{yy} = 47.7238$$

$$Y = \bar{y} + b(x - \bar{x})$$

$$Y = -4.6812 + 6.703x$$



### 4.3.3 Testing whether the line is an adequate representation of the data.

To facilitate the calculations we require a table with 10 columns:  $x$ ,  $n$ ,  $r$ ,  $Y$ ,  $P$ ,  $nP$ ,  $1-P$ ,  $nP(1-P)$ ,  $r-nP$  and  $\frac{(r-nP)^2}{nP(1-P)}$ . (See Table 4.3.3)

- (1) Choose  $\alpha$ , the level of significance of the test. In columns  $x$ ,  $n$ ,  $r$ , copy the values from the probit solution table. If the graphical probit solution was used, copy the  $Y$  column of that table as the  $Y$  column here. If the exact probit solution was used, use the  $\bar{y}$ ,  $\bar{x}$  and  $b$  obtained from the last iteration performed to compute, corresponding to each  $x$ ,

$$Y = \bar{y} + b(x - \bar{x})$$

and tabulate these values in the  $Y$  column here.

- (2) Look up  $\chi^2_{1-\alpha}$  for  $k - 2$  degrees of freedom in Table V.
- (3) Put  $z_p = Y - 5$ , and for each  $Y$ , using Table Ia, obtain the value of  $p$  corresponding to  $z_p$ .
- (4) Compute the required quantities in the last columns of the table.



$$(5) \quad \text{Obtain } \chi^2 = \sum_{i=1}^k \frac{(r_i - n_i p_i)^2}{n_i p_i (1 - p_i)} \quad , \quad \text{the sum of the}$$

values in the last column.

- (6) If  $\chi^2 \leq \chi^2_{1-\alpha}$ , decide there is no reason to believe that the line does not adequately represent the data. If  $\chi^2 > \chi^2_{1-\alpha}$ , decide that the straight line does not adequately describe the relation between stimulus and response. If a significant value of  $\chi^2$  is obtained, check to see whether an unusually large contribution to  $\chi^2$  comes from one class or a few classes with very small expected values, i.e., small  $nP$  or  $n(1-P)$ . If this is the case, one may want to combine several such classes (for details see [6]).

### Example

The test of the final probit equation is shown in Table 4.3.3. In Step (2), we find from Table V that  $\chi^2_{.95}$  for 8 d.f. = 2.73. In Step (5), the calculated  $\chi^2 = 2.31$ . Since this is not larger than  $\chi^2_{.95}$  we accept the fitted line.





Table 4.3.3 Test of Linearity - Final Probit Equation

$\log_{10} PV$	$x$	$n$	$r$	$Y$	$P$	$nP$	$1-P$	$nP(1-P)$	$r-nP$	$\frac{(r-nP)^2}{nP(1-P)}$
	1.0000	12	0	2.02	.00	.00				
	1.1761	12	0	3.20	.04	.48	.96	0.46	-.48	.50
	1.2430	12	1	3.65	.09	1.08	.91	0.98	-.08	.01
	1.3010	13	2	4.04	.17	2.21	.83	1.83	-.21	.02
	1.3522	10	3	4.38	.27	2.70	.73	1.97	.30	.05
	1.3979	13	6	4.69	.38	4.94	.62	3.06	1.06	.37
	1.4771	12	8	5.22	.59	7.08	.41	2.90	.92	.29
	1.5441	13	9	5.67	.75	9.75	.25	2.44	-.75	.23
	1.6021	13	11	6.06	.86	11.18	.14	1.57	-.18	.02
	1.6990	11	10	6.71	.96	10.56	.04	0.42	-.56	.75
	1.7780	11	11	7.24	.99	10.89	.01	0.11	+.11	.11

(43)

$$\chi^2 = 2.35$$

not significant

$$\chi^2_{.95}(8 \text{ d.f.}) = 2.73$$



#### 4.3.4 Using the Probit Regression Line for Prediction.

If a significant value of  $\chi^2$  is obtained using the procedure of 4.3.2, the formulas given in this section do not apply without adjustment (for details see [6]).

If no adjustment has to be made (i.e. the value of  $\chi^2$  is not significant) the following procedures describe how to use the probit regression line for estimation (including confidence interval estimates).

4.3.4.1 Level of stimulus  $x'$  at which a specified proportion  $P'$  of the individuals would be expected to respond.

##### Single estimate

- (i) Choose  $P'$
- (ii) Use Table Ib to find  $z_p'$ , corresponding to  $P'$
- (iii) Let  $Y' = 5 + z_p'$
- (iv) Compute  $x' = \bar{x} + (Y' - \bar{y})/b$  .

This  $x'$  is the value at which we would expect a proportion  $P'$  of the individuals to respond.

##### Confidence interval estimate

For a confidence interval estimate see [6]  
p. 63.



4.3.4.2 Level of stimulus  $x'$  at which 50% of the individuals would be expected to respond

From 4.3.4.1 we see that our estimate of  $\mu$ , the mean and the 50th percentile of the distribution of critical levels, is

$$m = \bar{x} + \frac{1}{b} (5 - \bar{y}) .$$

A  $(1-\alpha)$  confidence interval estimate of  $\mu$  is computed as follows.\*

i) Look up  $z_{1-\alpha/2}$  in Table Ib .

ii) Compute

$$s_m^2 = \frac{1}{b^2} \left[ \frac{1}{\sum nW} + \frac{(m-\bar{x})^2}{S_{xx}} \right]$$

iii) A  $(1-\alpha)$  confidence interval estimate of  $\mu$  is the interval from  $m - z_{1-\alpha/2} s_m$  to  $m + z_{1-\alpha/2} s_m$  .

---

\* This method is sufficiently good for most purposes. For an exact method see [6] p. 63.



(46)

4.3.4.3 Proportion of individuals which would be expected to respond at a specified level of stimulus.

The probit regression equation  $Y = \bar{y} + b(x - \bar{x})$  gives the expected value of  $Y(Y')$  at a specified value of  $x(x')$  .

The variance of  $Y$  ,  $s_Y^2$  , is given by the formula:

$$s_Y^2 = \frac{1}{\sum nW} + \frac{(x' - \bar{x})^2}{S_{xx}}$$

A  $(1-\alpha)$  confidence interval estimate for  $Y$  at a single specified value of  $x$  is given by the formula:

$$Y' \pm t_{1-\alpha/2} s_Y$$

If we want to make confidence interval statements for several values of  $x$  using the same fitted line, we must use instead the wider interval given by

$$Y' \pm \sqrt{2F} s_Y$$

as discussed in Part I, section 3.1.



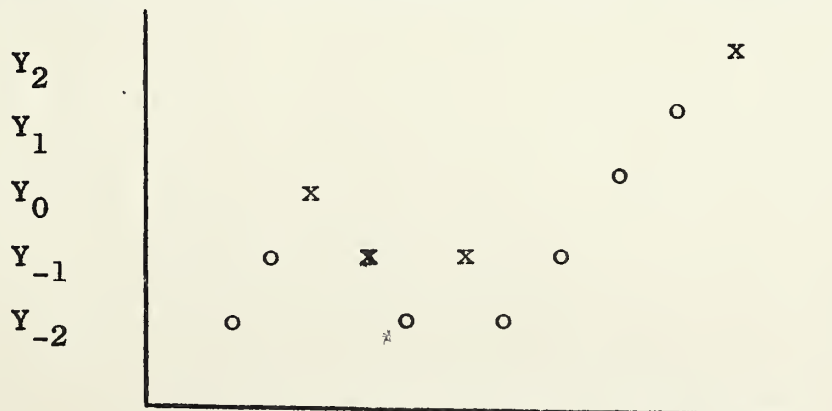


#### 4.4. The Up and Down Design

The "Up and Down" design is one of a class of designs called staircase methods because the test level for the next trial or group of trials depends on the results of the preceding trial or group of trials. In the "up and down" design only one object is tested at a time. Starting at a level where about 50% responses are expected, the test level is moved up one level after each non-response and down one level after each response. The experiment is concluded after a specified number of trials.

The use of the "up and down" design of course presumes that it is convenient to test one object at a time and all "staircase" methods presume that the results of test can be known immediately and that the test level can be adjusted quickly and easily.

If x's represent responses and 0's non-responses, then the pattern of the experiment looks like this:





The "up and down" design and its analysis will be described in detail below. It assumes normal distribution of the critical levels and in such a case will give a more accurate estimate of the mean (which is also the 50th percentile) than any other method described in this section. For further discussion of the up and down method, see [2, 4, 5]. Other staircase methods have been described in [10].



**Problem 4.4.** Estimate the mean of the distribution of critical levels when the "up and down" method of testing is used.

Initial guesses of the mean and standard deviation of the distribution of critical levels are required (Let  $x_0$  = guess for mean,  $d$  = guess for standard deviation). The method of estimation makes some allowance for a poor initial guess of the mean,  $x_0$ , and in fact is a particularly useful way of estimating the mean when the experimenter has little idea what the true mean is. So long as the initial guess for the standard deviation is between half and twice the true standard deviation of the distribution, the method of estimation described is appropriate.

Determine equally spaced test levels,  $\dots x_{-3}$ ,  $x_{-2}$ ,  $x_{-1}$ ,  $x_0$ ,  $x_1$ ,  $x_2$ ,  $x_3$ ,  $\dots$  so that the distance between two successive levels is  $d$ . ( $d = x_1 - x_0 = x_0 - x_{-1}$  etc.). The first object is tested at level  $x_0$ ; if it "responds", the second object is tested at level  $x_{-1}$ ; if it "does not respond", the second object is tested at level  $x_1$ . Similarly, each succeeding object is tested at the level one step below the level used in the preceding test if it resulted in a "response", at the level one step above the level used in the preceding test if it resulted in "no response".



Solution - Count the total number  $R = \sum r_i$  of responses and let  $N = \sum n_i$  be the total number of objects tested. If  $R \leq \frac{N}{2}$ , perform steps (1) through (6); if  $R > \frac{N}{2}$ , perform steps (1') through (6').

When  $R \leq \frac{N}{2}$ :

- (1) Prepare a table with columns headed  $y, r, j, j^2$ .
- (2) Let  $y_0$  = the lowest level at which a "response" occurred,  
 $y_1$  = the level one step above  $y_0$ ,  
 $y_2$  = the level two steps above  $y_0$ ,  
 $y_k$  = the highest level at which a "response" occurred.

Enter  $y_0, y_1, \dots, y_k$  in column  $y$ .

- (3) In column  $r$  corresponding to each  $y_j$ , enter  $r_j$  = the number of "responses" at level  $y_j$ .
- (4) Enter the numbers 0, 1, ...,  $k$  in column  $j$ .
- (5) Corresponding to each entry in column  $j$ , enter its square in column  $j^2$ .
- (6) Compute  
 $A = \sum j r_j$ , the sum of products of corresponding entries in columns  $r$  and  $j$ .  
 $B = \sum j^2 r_j$ , the sum of products of corresponding entries in columns  $r$  and  $j^2$ .





(51)

$$m = y_0 + d \left( \frac{A}{R} - \frac{1}{2} \right)$$
$$s = 1.620d \left( \frac{RB - A^2}{R^2} + .029 \right)$$

When  $R > \frac{N}{2}$  :

(1') Prepare a table with columns headed  $y$ ,  $n-r$ ,  
 $j$ ,  $j^2$  .

(2') Let  $y_0$  = the lowest level at which "no response"  
occurred,

$y_1$  = the level one step above  $y_0$  ,

$y_2$  = the level two steps above  $y_0$  ,

$y_k$  = the highest level at which "no response"  
occurred.

Enter  $y_0$  ,  $y_1$  , ...,  $y_k$  in column  $y$  .

(3') In column  $n-r$ , corresponding to each  $y_j$  , enter  
 $n_j - r_j$  = the number of "no response" at level  $y_j$  .

(4') Enter the numbers 0, 1, ...,  $k$  in column  $j$  .

(5') Corresponding to each entry in column  $j$ , enter  
its square in column  $j^2$  .



(6') Compute

$A = \sum j(n_j - r_j)$  , the sum of products of corresponding entries in columns  $n-r$  and  $j$  .

$B = \sum j^2(n_j - r_j)$  , the sum of products of corresponding entries in columns  $n-r$  and  $j^2$  .

$$m = y_o + d \left( \frac{A}{N-R} + \frac{1}{2} \right)$$

$$s = 1.62d \left\{ \frac{(N-R)B - A^2}{(n-R)^2} + .029 \right\}$$

$m$  is our estimate of the mean (and the 50th percentile) of the distribution of critical levels.

$s$  is our estimate of the standard deviation of the distribution of critical levels.



#### 4.5 Sensitivity Tests when the Stimulus Levels Cannot be Controlled.

The methods discussed in sections 4.2 through 4.4 have assumed that the stimulus levels can be preassigned and accurately controlled. Although this is the usual case in experimental work, there are times when conditions cannot be sufficiently well controlled to insure that the level used is exactly that which the experimenter intended to use. For example, he may intend to fire a group of 10 projectiles, each at a velocity of 2000 f/s but because of random variation in velocities for a fixed charge, the actual observed velocities range from 1975 to 2020 f/s.

In such a case, when the "level" used in the test can be directly measured, the experimenter has two choices for analyzing the data. If the actual levels used cluster so closely about the intended levels that

- (1) there is no overlapping between two of these clusters,  
and
- (2) the range of any cluster is so small the probability of "response" at any of the actual levels differs little from the probability of response at the intended level,



then the experimenter may simply assume that each test was conducted at the intended level, and use the methods already presented. However, if one or both of these conditions are not met, none of the methods described in this section are valid. Techniques for handling such data when the underlying distribution is "normal" are described in [7].





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